So far we have a formal way to describe how elements of a representation transform under an element of a group. But how can we build an invariant?

Given some thought, it might seem that combining two objects which transform "oppositely" would give an invariant. In fact this is exactly what we do!

We will take a cue from the familiar dot product, i.e. $\vec{v}\cdot\vec{w} = \#$ or $(v_1\ v_2\ v_3)\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = v_1w_1 + v_2w_2 + v_3w_3$

For any matrix representation $r$ we can form the <u>dual representation</u> $\tilde{r}$ as follows:

If $A \in G$ then $r \to Ar$ , $\tilde{r} \to (A^{-1})^T \tilde{r}$.

Then if we form $\tilde{r}^T r \to (A^{-1^T}\tilde{r})^T Ar = \tilde{r}^T A^{-1^{T^T}} Ar = \tilde{r}^T A^{-1} A r = \tilde{r}^T r$

So far I simply wrote down the dual representation. But we could ask if there is a systematic way to construct the dual $\tilde{r}$ if we are given $r$. For many cases this can be done if we are given a metric.

A metric $g$ a map from an element of a representation $r$ to a corresponding element of the dual representation $\tilde{r}$, i.e. $\tilde{r} = g r$. The metric will always be a symmetric matrix.

Based on this definition let's see what $\tilde{r}^T r = $ invariant implies about the metric $g$.

$$\tilde{r}^T r = (gr)^T r = r^T g^T r = r^T g r \longrightarrow (Ar)^T g A r = r^T A^T g A r \quad \text{for some } A \in G$$

since $g$ is symmetric

then $= r^T g r$ if $\boxed{A^T g A = g}$

The important lesson here is: If we have some representation $r$ of a group $G$, then forming a dual representation $\tilde{r}$ with the metric $g$ will give an invariant $\tilde{r}^T r$ if $A^T g A = g$ for $A \in G$.

And so we found that given a group $G = \{\Lambda_1, \Lambda_2, \Lambda_3 \cdots\}$ and a linear representation $r$ (column vectors), we can form a dual representation $\tilde{r}$ such that $\tilde{r}^T r = $ invariant using a metric $g$ that satisfies $\underbrace{\Lambda_i^T g \Lambda_i = g}_{\text{We can work this 2 ways}}$ for any $\Lambda_i \in G$.

Given $G = \{\Lambda_i\}$ find a metric $g$

We'll start w/ an example of this.

Given the metric $g$ find the transformations $G = \{\Lambda_i\}$

Then apply this to define SR

Rotations in 3D: We focus on transformations of coordinates, not objects, i.e. passive rotations.

Form a compact, continuous, non-abelian group. We will denote rotations by R,

so $G = \{ \underbrace{R_x(\theta),\ R_y(\phi),\ R_z(\psi)}_{\text{continuous parameters}} \}$

You probably know what these look like, but let's explore their mathematical structure.

Consider 2 objects in space whose positions are specified in a given rectangular coordinate system by $(x_A, y_A, z_A)$ and $(x_B, y_B, z_B)$. We have a sense that if we knew the "distance" between these two objects, then this should be invariant under a change of coordinates (well at least in this particularly simple case!)

The distance between these two points can be expressed as:

$$\Delta s = \sqrt{(\underbrace{x_A - x_B}_{\Delta x})^2 + (\underbrace{y_A - y_B}_{\Delta y})^2 + (\underbrace{z_A - z_B}_{\Delta z})^2} = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$$

Let's deconstruct this a bit. Consider $\Delta s^2 = (\Delta x \ \ \Delta y \ \ \Delta z) \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$

Now we know that if change to a new (rotated) coordinate system, then all of the

$$\begin{matrix} (x_A, y_A, z_A) \\ (x_B, y_B, z_B) \end{matrix} \rightarrow \begin{matrix} (x'_A, y'_A, z'_A) \\ (x'_B, y'_B, z'_B) \end{matrix} \Rightarrow \Delta x, \Delta y, \Delta z \rightarrow \Delta x', \Delta y', \Delta z'$$ will change, but $\Delta s^2$ will not.

If our rotation is represented by a linear operator then:

$$\begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} \rightarrow \begin{pmatrix} \Delta x' \\ \Delta y' \\ \Delta z' \end{pmatrix} = R \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} \Rightarrow (\Delta x \ \Delta y \ \Delta z) \rightarrow (\Delta x' \ \Delta y' \ \Delta z') = \left[ R \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} \right]^T$$

Then: $$\Delta s^2 = (\Delta x \ \Delta y \ \Delta z) \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} = (\Delta x' \ \Delta y' \ \Delta z') \begin{pmatrix} \Delta x' \\ \Delta y' \\ \Delta z' \end{pmatrix}$$

$$= (\Delta x \ \Delta y \ \Delta z) R^T R \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$

So invariance of $\Delta s^2$ requires $R^T R = \underline{I}$.

So invariance of $\Delta s^2$ requires $R^T R = I$.

Comparing to $\Lambda^T g \Lambda = g$ we see that in the case of 3D rotations in Euclidean space $g = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} = I$

So we should understand the distance expression as: $\Delta s^2 = (\Delta x \;\; \Delta y \;\; \Delta z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$

Now we can understand why we called $g$ a metric. For spacetime transformations it plays a crucial role in defining distance. In fact it will help us tremendously to think of coordinates as labels distinguishing points in space (time) with no intrinsic notion of distance. The distance we encode in the metric.

To make this very concrete consider 2D polar coordinates $(r, \theta)$ then (infinitesimally)

$$ds^2 = dr^2 + r^2 d\theta^2 = (dr \ d\theta) \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \begin{pmatrix} dr \\ d\theta \end{pmatrix} = dr^2 + r^2 d\theta^2$$

If you tried to just use the coordinate differences you might write $ds^2 = dr^2 + d\theta^2$ which is not correct!

We now have sufficiently powerful tools to state the premise of Special Relativity:

SR - Physical laws should not change under transformations (continuously connected to $I$) which preserve spacetime intervals of the form $\Delta s^2 = -c^2 \Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$.

This immediately implies invariance under translations $t \to t' = t + \Delta t$

$x \to x' = x + \Delta x$

etc.

Additionally since: $\Delta s^2 = (c\Delta t \;\; \Delta x \;\; \Delta y \;\; \Delta z) \underbrace{\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}}_{\equiv \, \eta} \begin{pmatrix} c\Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$

$\eta$ - the metric of Minkowski space (in rectangular coordinates)

Then we know that any $\Lambda$ satisfying $\Lambda^T \eta \Lambda = \eta$ will leave the interval invariant.

<u>All</u> of the love in SR flows from this invariance principle.

So what are the $\Lambda$'s? First of all there an infinite # since this is a continuous group. But just like w/ rotations, we should be able to organize them into a small set of independent transformations each labelled by a continuous parameter.

In fact if we are savvy, we can closely follow the familiar example of rotations.

First: $R_x(\theta), R_y(\theta), R_z(\psi)$ seems to imply that we just need to add something like $R_t(\beta)$, but this is actually wrong! The problem is an "accident" of 3D where there is exactly one axis normal to any plane. Hence x-axis $\leftrightarrow$ yz-plane are interchangeable.
y-axis $\leftrightarrow$ zx-plane
z-axis $\leftrightarrow$ xy-plane

In 4D this is not the case. So what do we count, axes or planes? Well just think about rotations in 2D (where there is no $\perp$ axis!).

Counting planes we find that in N dimensions there are $\binom{N}{2} = \frac{N!}{(N-2)!\,2!} = \frac{1}{2}N(N-1)$ choices of plane.

So in 3D this gives 3 planes xy, yz, zx
But in 4D we have 6 planes xy, yz, zx, xt, yt, zt

You can probably guess the $\Lambda$ which corresponds to a spatial plane, e.g. $\Lambda_{xy}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Checking: $\Lambda_{xy}^T \eta \Lambda_{xy} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \cos^2\theta+\sin^2\theta & \cos\theta\sin\theta-\cos\theta\sin\theta & 0 \\ 0 & \cos\theta\sin\theta-\cos\theta\sin\theta & \cos^2\theta+\sin^2\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \eta$

For the spatial planes this hinged on $\cos^2\theta + \sin^2\theta = +1$ which we could anticipate from $\Delta x^2 + \Delta y^2$.
For a space-time plane, e.g. $\Lambda_{xt}$ we need something else. Given $-c^2\Delta t^2 + \Delta x^2$ we might consider $-\sinh^2\phi + \cosh^2\phi = 1$

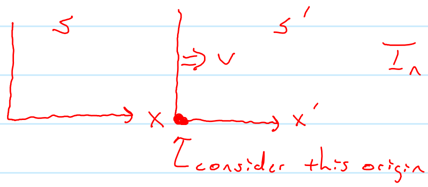
So let's try $\Lambda_{xt} = \begin{pmatrix} \cosh\phi & -\sinh\phi & & \\ -\sinh\phi & \cosh\phi & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

Checking: $\Lambda_{xt}^T \eta \Lambda_{xt} = \begin{pmatrix} -\cosh^2\phi+\sinh^2\phi & \cosh\phi\sinh\phi-\cosh\phi\sinh\phi & & \\ \cosh\phi\sinh\phi-\cosh\phi\sinh\phi & -\sinh^2\phi+\cosh^2\phi & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

Now I know what you're thinking: $\Lambda_{xt}$ W.T.F.

Consider: $\begin{pmatrix} c\Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} \rightarrow \begin{pmatrix} c\Delta t' \\ \Delta x' \\ \Delta y' \\ \Delta z' \end{pmatrix} = \Lambda_{xt} \begin{pmatrix} c\Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} = \begin{pmatrix} \cosh\phi\, c\Delta t - \sinh\phi\, \Delta x \\ -\sinh\phi\, c\Delta t + \cosh\phi\, \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$

Now consider:




In $S'$ the origin is at rest so $\Delta x' = 0$ so from above $\Delta x' = 0 = -\sinh\phi\, c\Delta t + \cosh\phi\, \Delta x$

or $\frac{\Delta x}{\Delta t} = c\tanh\phi$

But in $S$ (where $x, t$ are used) the origin is moving w/ $v_x = v$ so $\frac{\Delta x}{\Delta t} = c\tanh\phi = v$

$\tanh\phi = \frac{v}{c}$

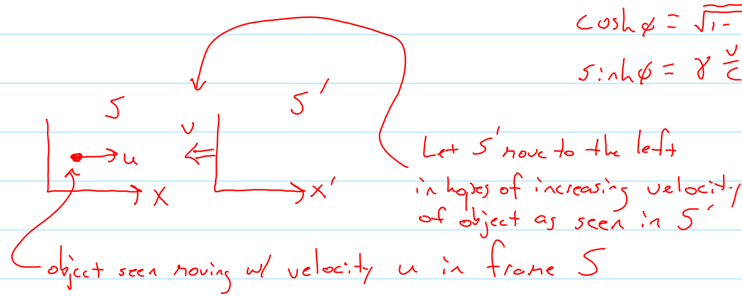
$\Downarrow$

$\cosh\phi = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \equiv \gamma$

$\sinh\phi = \gamma \frac{v}{c}$

Then: $\begin{pmatrix} c\Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} \rightarrow \begin{pmatrix} c\Delta t' \\ \Delta x' \\ \Delta y' \\ \Delta z' \end{pmatrix} = \begin{pmatrix} \gamma(c\Delta t - \frac{v}{c}\Delta x) \\ \gamma(\Delta x - v\Delta t) \\ \Delta y \\ \Delta z \end{pmatrix}$

Let's see how another important result follows:




From above (with $v \rightarrow -v$):

$\frac{\Delta x'}{\Delta t'} = \frac{\gamma\Delta x + \gamma v \Delta t}{\gamma\Delta t + \gamma\frac{v}{c^2}\Delta x}$ multiplying by $\frac{\frac{1}{\Delta t}}{\frac{1}{\Delta t}} \Rightarrow \frac{\Delta x'}{\Delta t'} = \frac{\gamma\frac{\Delta x}{\Delta t} + \gamma v}{\gamma + \gamma\frac{v}{c^2}\frac{\Delta x}{\Delta t}} = \frac{\frac{\Delta x}{\Delta t} + v}{1 + \frac{v}{c^2}\frac{\Delta x}{\Delta t}}$

But $\frac{\Delta x}{\Delta t} = u$ so $\frac{\Delta x'}{\Delta t'} = u' = \frac{u+v}{1+\frac{vu}{c^2}}$ which is the velocity addition formula for SR!